Large deviation theory for coin tossing and turbulence

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Large deviations play a significant role in many branches of nonequilibrium statistical physics. They are difficult to handle because their effects, though small, are not amenable to perturbation theory. Even the Gaussian model, which is the usual initial step for most perturbation theories, fails to be a starting point while discussing intermittency in fluid turbulence, where large deviations dominate. Our contention is: in the large deviation theory, the central role is played by the distribution associated with the tossing of a coin and the simple coin toss is the "Gaussian model" of problems where rare events play significant role. We illustrate this by applying it to calculate the multifractal exponents of the order structure factors in fully developed turbulence.

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Fully developed turbulence in fluid [1–13] is basically multifractal—a term coined for the very first time in a paper by Benzi *et al.* [14]. The multifractal formulation was documented in great detail by Parisi and Frisch [15] (also see [16]). Among the precursors to the multifractal formulation are the works by Kolmogorov [1], Obukov [2], and Mandelbort [17]. It is widely accepted that the concept of multifractality, technically speaking, stands on the shoulders of the large deviation theory [18–20]. However, one should be careful enough while using large deviation theory in turbulence for deriving exponents of the velocity structure functions because Frisch *et al.* [21] have shown that one must make use of "refined" large deviation theorem [22] as had already been anticipated by experimentalists [23] on the basis of a normalization requirement.

The scaling exponents of the structure functions of the velocity field difference between two points in a turbulent fluid are known to be related nonlinearly on the orders of the structure functions. In the model of Kolmogorov and Obukhov, the nonlinearity was for the first time, attributed to large fluctuations in the velocity difference, which in turn was supposed to be triggered by large fluctuations in the dissipation rate coarse grained at the same scale. Since then, a number of models have been proposed to understand the essential features of this "intermittent" behavior. Among these, there is the multifractal model referred to above, which interprets the experimental results by assuming multifractal nature for the probability distribution function of the energy dissipation rate. This model does not make predictions, rather interprets the exponents of the scaling laws for the coarse grained dissipation field in terms of a singularity spectrum, defined as the Legendre transform of the exponents.

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Within the paradigm of multifractal model of turbulence, where one assumes that the velocity has a local scale invariance, it is not quite hard to find phenomenological models that can faithfully enough reproduce anomalous scaling exponents. However, what we have focused on in this paper is quite different than what the usual research on multifractality is all about. We emphasize that the rare events present in the distribution of energy dissipation in real space, when "mapped" appropriately on the phenomenon of large deviations found in simple coin toss, are enough to yield anomalous exponents. Quite interestingly, here one does not has to fall back on any explicit model of energy cascade, e.g., random multiplicative model [11], etc.

In what follows in this paper shortly, we propose an approach that allows us to construct a simple (tunable) parameter dependent model that has the amazing potential of yielding quantitative results. While constructing such a model, we mainly rely on the observation that the concerned physical process (here, turbulence) has certain relevant rare events present in it. In the case of turbulence, our model's success can be interpreted as the reconfirmation that the phenomenon of multifractality owes itself to the rare events present in the distribution of energy dissipation. To be precise, to construct our model for turbulence, we have assumed that square of one-dimensional velocity gradient [scaled appropriately as shown later in Eqs. (6)–(8)] minus the expected mean of the energy dissipation rate is a bounded, independent and identically distributed random variable. On the face of the fact that velocity field is random in a turbulent flow, this assumption is not very artificial. We want to stress here that our model does not stand against the multifractal model of turbulence. Rather, our model supplements the multifractal model and also, uses its results for benchmarking. Our framework, on the top of it, has the advantage of being applicable to any other physical phenomenon where there is no known multifractal (or any other) explanation of the results due to the presence of rare events therein. Our methodology may look like a "black-box," but the point is that it is capable of delivering genuine results that can be experimentally verified.

That this approach can be very helpful in the subjects unrelated to turbulence, can be carefully illustrated by successfully applying the idea on the problem of Jarzynski equality [24,25] (to be reported elsewhere in details) to obtain work moments that are seen to compare favorably with the experimentally obtained moments [26]. Jarzynski equality is an exact form among many fluctuation theorems [27] that form a very important part of nonequilibrium statistical mechanics. One might recall that the rare events present in the dissipated work enter Jarzynski equality. These rare events correspond to the negative values of the dissipated work which constitute a transient violation of the second law of thermodynamics. Though Jarzynski equality and turbulence are fundamentally different, our approach can be applied to both of them (and many other such phenomena)—the only connecting element being the presence of rare events. While the rare events shaping Jarzynski equality are the transient violations of the second law of thermodynamics, the rare events which will be considered in turbulence has to do with the occasional large fluctuations in the energy dissipation rate which normally would stay very close to the mean value. The link between the two very different phenomena is the occurrence of the rare events.

As mentioned earlier, the high Reynolds number turbulence remains the prime age old problem dominated by rare events, which still eludes a satisfactory theoretical understanding. Before we plunge into the problem of modeling intermittency in a turbulent fluid, let us begin by briefly reviewing the large deviation theory in the context of a coin-toss experiment. Suppose we have a biased coin, such that for each toss the probability of obtaining "head" is "p." If we assign the value 1 to the outcome head (each outcome is denoted by X_i where $i=1,2,\ldots,$) and 0 to the outcome "tail," then the mean after N trials is

$$M_N = \frac{1}{N}\sum_{i=1}^{N} X_i. \tag{1}$$

As $N\to\infty$, it is expected that $M_N\to p$. The question is: for large N, what is the probability that M_N differs from p by at least x (where x is any preassigned fraction less than unity)? The meaning of large deviation is that however large N may be this probability is nonzero and if the X_i's are bounded, independent, and identically distributed random variables, then Cramers' theorem asserts that the tail of the probability distribution of X_i is given by

$$\left.\begin{array}{ll} P(M_N > x) \approx e^{-NI(x)} & \text{for } x > p \\ P(M_N < x) \approx e^{-NI(x)} & \text{for } x < p \end{array}\right\}. \tag{2}$$

To apply this result in different disciplines of statistical physics, we require $P(M_N\approx x)$ and it is Varadhan's theorem that ensures that the sequence M_N itself satisfies a large deviation principle, i.e., $P(M_N\approx x)\sim e^{-NI(x)}$. For the coin toss under consideration, Chernoff's formula gives the rate function $I(x)$ as follows:

$$I(x) = x\ln\frac{x}{p} + (1-x)\ln\frac{1-x}{1-p}, \tag{3}$$

and this is the central result that we will use.

Turning to turbulence, in 1941 Kolmogorov [28] invoked the concept of Richardson's cascade [29] of eddies to propose a phenomenological model (K41) for three-dimensional incompressible turbulence at high Reynolds number. Even today, this is the cornerstone of our understanding of turbulence. Understanding turbulence is understanding the small scale behavior of the velocity structure function $S_q(l)$, where $S_q(l)\equiv\langle|\Delta\vec{v}.(\vec{l}/|l|)|^q\rangle$, with $\Delta\vec{v}\equiv\vec{v}(\vec{r}+\vec{l})-\vec{v}(\vec{r})$ and "l" is a distance which is short compared to macroscopic length scales like the system size but is large compared to molecular scale where viscous dissipation takes place. The angular bracket denotes ensemble average (i.e., average over different values of "$\vec{r}$"). The observation is that $S_q(l)$ has a scaling behavior l^{ζ_q} where l is in the range indicated (so-called inertial range). Finding ζ_q can be described as the holy grail of turbulence. K41 gives $\zeta_q=q/3$—a result which is exact for $q=3$ and very close to experimental findings for low value of q. There is systematic departure from $q/3$ at relatively higher values of q. This is the phenomenon of intermittency. Of particular interest is the case $q=6$. Since $|\Delta v|^3/l$ is a measure of the local energy transfer rate (same as energy input and energy dissipation rate in K41 and thus a constant), we expect $\zeta_6=2$. The deviation $2-\zeta_6$ is thus a very sensitive quantity and is often singled out for special treatment. The exponent $\mu=2-\zeta_6$ is formally called the intermittency exponent and the experimental measurements agree on a value 0.2 for μ. It can be viewed as the co-dimension of dissipative structures.

The model of intermittency is usually constructed on a phenomenological basis by thinking of various ways of modifying the Richardson's cascade picture. The β model, the bifractal model, and the multifractal model all belong to this class. The crucial hypothesis is that the daughter eddies produced from the mother eddies are not space filling and the active part of space is in general a multifractal. The velocity field has different scaling exponents on different fractal sets that form the multifractal structure. These scaling exponents can, in principle, yield ζ_q. This multifractality can also be defined and measured in terms of the fluctuations of the local dissipation rate rather than in terms of the fluctuations of the velocity increments Δv. The key element that is needed to define multifractality in terms of dissipation is the local space average of energy dissipation over a ball of radius l centered around a point at $\vec{r}:\varepsilon_l(\vec{r})\equiv\frac{3\nu}{8\pi l^3}\int_{|\vec{r}'-\vec{r}|<l}d^3\vec{r}'\sum_{i,j}[\partial_j v_i(\vec{r}')+\partial_i v_j(\vec{r}')]^2$. If the dissipation is multifractal, moments of ε_l follow a power law behavior at small l, i.e., $\langle\varepsilon_l^q\rangle\sim l^{\tau_q}$. Kolmogorov's refined similarity hypothesis relates the statistical properties of fluctuation of velocity increment to those of the space averaged dissipation and yields: $\zeta_q=\frac{q}{3}+\tau_{q/3}$. We now carry out the usual speculation that since the higher order velocity structure factors differ most strongly from K41, then the probability distribution for the velocity increments must differ most strongly from that appropriate to K41 in the tail of the distribution. The tail

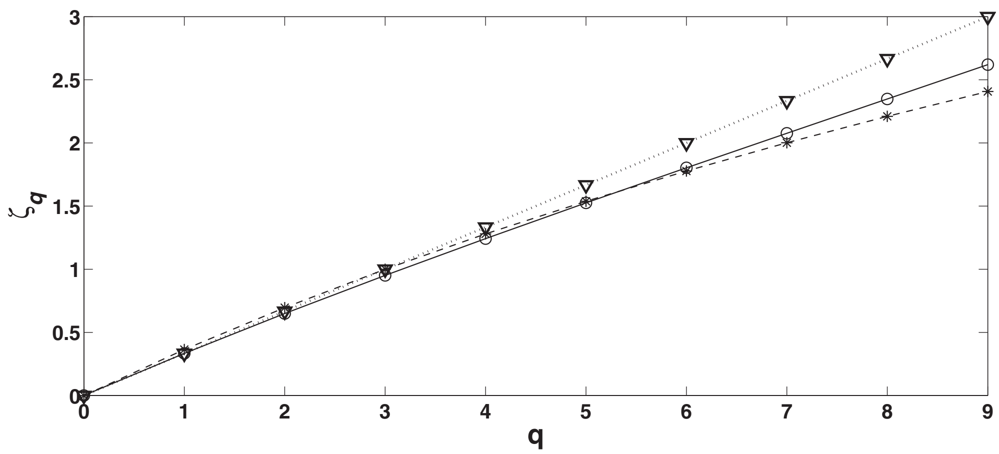

FIG. 1. ζ_q vs q curve in fully developed fluid turbulence. The dashed line joining the asterisks is the celebrated She-Leveque scaling law. The circles joined by the solid line denote the values of ζ_p (for corresponding q) as obtained by dint of the model proposed herein. To appreciate the convexity of the aforementioned curves, a dotted line joining triangles, in accordance with the classical linear Kolmogorov prediction, has also been plotted. For every q, first $\langle|\varepsilon_l-\varepsilon|^q\rangle$ vs N is plotted in log-log scale using the data yielded during the numerical integration of Eq. (9) and then the observation that for $N=30$ to 60, we get a fairly straight line leads us to attempt fitting the range linearly. The process gives a value for τ_q. The relation $\zeta_q=q/3+\tau_q$, then, tells us what the corresponding value is for ζ_q. One can see the fit is remarkable. There is room for improvement in extending the inertial range and in getting better fit for higher ζ_q's. As mentioned in this paper, the form of Z_T is crucial.

of a distribution involves rare events and this is how the theory of large deviations enters the picture. Following Landau's observation on K41 [30], Kolmogorov [1] and Obukhov [2] introduced fluctuations in the dissipation rate. Careful experiments revealed the existence of these fluctuations. The fluctuations, however, occur rarely and these are the rare events of turbulence. This allows us to establish a quantitative bridge between turbulence and theory of large deviations. The above discussion parallels what can be found in [11] (in particular Sec. 8.6.4). The quantitative development in [11] thereafter focuses on a particular model which has been taken to be a random cascade model. What is clearly shown over there is the fact that the general random cascade model, which exhibit multifractal behavior, is related to the large deviation theory in which the function $I(x)$ in Eq. (2) is just the $f(\alpha)$ function of a multifractal distribution. What we will do in the following will be to exploit the above discussion by (a) treating the dissipation at different points inside a coarse graining volume as independent random variables and (b) using a suitable mapping which makes the result from Cramers' theorem for the coin toss, relevant.

More than a decade ago, Stolovitzky and Sreenivasan [12], in a somewhat different approach, tried to validate refined similarity hypothesis by viewing turbulence as a general stochastic process (fractional Brownian motion to be precise). While this was a very significant achievement, there was a shortcoming in that the theory ruled out the existence of correlation functions like S_3. It indeed is surprising since the readers may know that the only exact nontrivial result existing in the theory of turbulence is Kolmogorov law: $S_3(l)=-\frac{4}{5}\varepsilon l$. However, as we shall note, their approach allows us to make direct contact with the terms of large deviation that signify the occurrence of rare events. It can be observed that deviation of ε_l from the expected mean ε plays the role of M_N of Eq. (1) and it is what we are interested in. As $l\to\infty$, this deviation variable has a distribution according to the role of Eq. (2). We hope a simplification: the $\varepsilon_l-\varepsilon$ can range from large negative to large positive values. We bring the range between 0 and 1 by defining a variable as

$$Z_T(\varepsilon_l) \equiv \frac{1}{2}\left[1+\tanh\left(\frac{\varepsilon_l-\varepsilon}{\Xi}\right)\right], \tag{4}$$

where Ξ is a constant with dimension of ε. We now make the drastic assumption that since $\varepsilon_l-\varepsilon$ is a rare event, the distribution of Z_T can be considered similar to that for the coin toss with a biased coin and accordingly, we can hypothesize that

$$P(Z_T) \propto e^{-NI(Z_T)}. \tag{5}$$

Here, N is number of random variables. This simple model yields value of $\mu\approx 0.16$, which is quite close to the presently accepted value. It can be taken as an *a posteriori* justification for our seemingly bold above-proposed postulate regarding the distribution of $Z_T(\varepsilon_l)$. Also, a ζ_q vs q plot has been obtained that is not only convex but also follows She-Leveque scaling [13] faithfully enough for a model as simple as this (please refer to Fig. 1). In what follows, we describe how these results are arrived at.

The one-dimensional velocity derivative can be use to express the global average of the full energy dissipation if local isotropy exists [31,32]. The velocity increment is given by

$$\Delta v(l) = \int_r^{r+l} \frac{dv}{dr}dr, \tag{6}$$

and *ergo*, the energy dissipation rate is

$$\varepsilon(l) = \frac{15\nu}{l}\int_r^{r+l}\left(\frac{dv}{dr}\right)^2 dr. \tag{7}$$

If we define $D_i \equiv \frac{dv}{dr}|_i[\frac{\eta\sqrt{15\varepsilon}}{(\eta\varepsilon)^{1/3}}]$ and $N \equiv \frac{l}{K\eta}$ (where, η is Kolmogorov scale, $(\eta\varepsilon)^{1/3}$ is Kolmogorov velocity scale and K is the number of Kolmogorov scales over which one obtains smoothness), then Eq. (7) may be rewritten, upon discretization, as

$$\varepsilon_l - \varepsilon = \frac{1}{N}\sum_{i=1}^{N} Y_i. \tag{8}$$

Here, $Y_i \equiv D_i^2 - \varepsilon$. The link between the phenomenology of turbulence and the theory of large deviation comes from the above equation where we assume each Y_i to be an independently distributed random variable. It is the possibility that the experimental average as expressed in Eq. (8) can show significant departure from zero for large $N(N \propto l)$ leads to the l dependence of the powers of the deviation $\epsilon_l - \epsilon$ and thus to the multifractality of turbulence. As we would like to emphasize that this is a very direct way of quantifying the link between turbulence phenomenology and the theory of large deviations with the help of the binomial distribution. The fact that the results are not as good as She and Leveque [13] is merely a statement of the fact that the assumed binomial distribution is not the most accurate one. Here, we have assumed the relation (8) to be the parallel of Eq. (1). Owing to the contraction principle, the rate functions for $\varepsilon_l - \varepsilon$ and $Z(\varepsilon_l)$ are same. Thus, using Eqs. (3)–(5), we can write

$$\langle|\varepsilon_l - \varepsilon|^q\rangle = \left|\frac{\Xi}{2}\right|^q \left[\frac{\int_0^1 \left|\ln\left(\frac{x}{1-x}\right)\right|^q \left\{\left(\frac{p}{x}\right)^x\left(\frac{1-p}{1-x}\right)^{1-x}\right\}^N dx}{\int_0^1 \left\{\left(\frac{p}{x}\right)^x\left(\frac{1-p}{1-x}\right)^{1-x}\right\}^N dx}\right]. \tag{9}$$

We assume that to the leading order $\langle|\varepsilon_l - \varepsilon|^q\rangle \sim l^{\tau_q}$. By trial and error, we fix the inertial range as $N=30$ to 60 and calculate numerically $\mu(=-\tau_2)=0.16$. Similarly, we calculate $\zeta_q(=q/3+\tau_{q/3})$ for various q. Note that to obtain the numerical solution for the integrals in Eq. (9), we have dropped the diverging terms from the finite series that represent the integrands as they are suitably discretized for their evaluation by Simpson's one-third rule.

Our model's inherent bias for the value 0.26 for the parameter p in order to closely mimic the realistic turbulent fluid's scaling properties would seem so natural when it is compared with a particular successful multifractal cascade model [33] based on a generalized two-scale Cantor set. In that model, as the eddies breakdown into two new ones, the flux of kinetic energy into the smaller scales is hypothesized to be dividing into nonequal fractions $p=0.3$ (quite close to our value of $p=0.26$) and $1-p=0.7$. It could fit remarkably well the entire spectrum of generalized dimensions [34] and (equivalently) the singularity spectrum (the so-called $f-\alpha$ curve [35]) for the energy dissipation field in many a turbulent flow.

In closing, we would like to point out the simplicity of biased coin-toss models and its reasonably astonishing success in predicting μ reduces the need for more complicated models. We believe just by being able to find a more appropriate function Z_T, we can make big leaps in the rather complex theory of turbulence. One should note that the refined large deviation theorem, which implies the presence of the factor $\sqrt{N}$ in the probability density, has no extra effect on the results derived herein using large deviation theorem. As readers must have appreciated, we could derive results concerning anomalous exponents (showcasing intermittency in turbulence) merely by focusing on the presence of rare events in the distribution of energy dissipation rate and by mapping them appropriately on the phenomenon of large deviations found in simple coin toss. Therefore, it is in accordance with our contention that the simple coin toss is the "Gaussian model" for the problems where rare events play significant role, Within this very framework, we hope to model various other physical phenomena that are dominated by rare events; after all, now we have a working approach to arrive at quantitative results for such processes that cannot be usually solved otherwise.

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